Selected Exercises

Nuclear Forces PHY989

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Fall semester 2017

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Exercise 1: Questions on the Overview of QCD

This collection of problems contain short exercises and discussion questions on QCD.

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a) With respect to what scale(s) are the c, b, t quarks called heavy?

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b) Have you heard about the s quark before? If yes, in what context?

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c) A possible way to *see* quarks and gluons is in jets. What happens in these events?

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d) Using the Particle Data Group website, discuss which properties of the neutron and proton are similar and what are differences? What about for the three pions?

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e) Which is more important in making a neutron more massive than a proton: the light quark mass difference or the electromagnetic contribution? Or do you think such considerations are too simplistic?

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f) What is the evidence for *spontaneous* chiral symmetry breaking in

- the mass spectrum of pseudoscalar ($J^\pi = 0^-$) mesons;
- the mass spectrum of vector and axial vector ($J^\pi = 1^\mp$) mesons?

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g) What is the evidence for *explicit* chiral symmetry breaking in the spectrum of pseudoscalar ($J^\pi = 0^-$) mesons?

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h) If you and your friend each do a QCD calculation with the same diagrams but use α_s at different scales, will you get the same answer? If not, how could that happen?

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i) Does the running coupling in QCD mean that the QCD Hamiltonian is not unique? Would you say that if you used α_s at two different scales that you were using two different Hamiltonians?

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j) If the neutron lifetime is so short, why are there *any* stable nuclei?

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k) One observes a marked resonance when a π^+ pion is scattering off a proton. Which baryon does this correspond to and at which energy of the π^+ does this occur (the proton is at rest)?

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l) At sufficient energy in proton-proton collisions it is possible to create a pion, $p + p \to p + n + \pi^+$. At which energy in the center-of-mass frame does pion production start?

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Exercise 2: Basic Scattering Theory

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a) We typically use units in which $\hbar = c = 1$ and express quantities as powers of MeV or fm or both, using $\hbar c \approx 197.33$ MeVfm to convert between them. If we take for the nucleon mass $M_N = 939\ \text{MeV}/c^2$, what is $\hbar^2/M_N$ numerically in terms of MeV and fm?

Hint. Hint: This should be almost immediate if you insert the right factors of c.

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b) For the scattering of equal mass (nonrelativistic) particles, if the laboratory energy E_{lab} is related to the magnitude of the relative momentum k_{rel} (i.e., the momentum each particle has in the center-of-mass frame) by $E_{\text{lab}} = C k_{\text{rel}}^2$, what is C? If the mass is $M_N = 939$ MeV, what is the value of C in MeVfm^2?

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c) We write the partial-wave momentum space Schroedinger equation (see Lecture notes) as

$$\frac{k^2}{2\mu}\langle klm|\psi\rangle + \frac{2}{\pi}\sum_{l'm'}\int_0^\infty dk'\, k'^2\, \langle klm|V|k'l'm'\rangle\langle k'l'm'|\psi\rangle = E_k\langle klm|\psi\rangle\ ,$$

what are the units of $V_{ll'}(k,k') \equiv \langle klm|V|k'l'm\rangle$? In coordinate space the potential is local, $V(r)$, with units of MeV, and k is given in inverse fm. If you see a plot in a journal article of $V_{ll'}(k,k')$ with units of fm, how would you convert it to the units you just found?

Hint. Hint: use the results from the first exercise here.
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d) In Figure 18 of the review by Scott Bogner *et al.*, Prog. Nucl. Part. Phys. **65**, 94 (2010) the momentum-space matrix elements of different chiral effective field theory potentials are given in units of fm. Consider the value at zero relative momenta. $\tilde{C}_{^1S_0}$, see Eq. (2.5) and the article by Epelbaum *et al.* in GeV^{-2}. How do you convert to fm units? Do the values for the matrix elements then match?
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e) What do *on-shell* and *off-shell* mean in the context of scattering?
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f) Under what conditions is a partial-wave expansion of the potential useful?
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g) Derive the standard result:

$$\frac{e^{i\delta_l(k)}\sin\delta_l(k)}{k} = \frac{1}{k\cot\delta_l(k) - ik}$$

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h) Given a potential that is not identically zero as $r \to \infty$ (e.g., a Yukawa), how would you know in practice where the asymptotic (large r) region starts?
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i) What is the physical interpretation of the relation between the (partial-wave) S-matrix and the scattering amplitude? (Note that $S_l(k) = 1 + 2ikf_l(k)$.)

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Exercise 3: More on the Lippmann-Schwinger equation
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a) Using the Schrödinger equation for the scattering of two particles with mass m,

$$(H_0 + V)|\psi_E\rangle = E|\psi_E\rangle\ ,$$

where H_0 is the free Hamiltonian, show that the Lippmann-Schwinger equation for the wave function,

$$|\psi_E^{\pm}\rangle = |\phi_k\rangle + \frac{1}{E - H_0 \pm i\epsilon} V|\psi_E^{\pm}\rangle\ ,$$

is satisfied. Here $E = k^2/m$ and the plane wave state satisfies $H_0|\phi_k\rangle = E|\phi_k\rangle$. Why do you need the $\pm i\epsilon$?

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b) We can define the T-matrix on-shell as the transition matrix that acting on the plane wave state yields the same result as the potential acting on the full scattering state. That is, $T^{(\pm)}(E = k^2/m)|\phi_k\rangle = V|\psi_E^{\pm}\rangle$. What does it mean that the T-matrix is *on-shell*? (This is a really quick question!)

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c) Show that matrix elements of the T-matrix satisfy the Lippmann-Schwinger equation

$$\langle \mathbf{k}'|T^{(\pm)}(E)|\mathbf{k}\rangle = \langle \mathbf{k}'|V|\mathbf{k}\rangle + \int d^3p \, \frac{\langle \mathbf{k}'|V|\mathbf{p}\rangle\langle \mathbf{p}|T^{(\pm)}(E)|\mathbf{k}\rangle}{E - \frac{p^2}{m} \pm i\epsilon}.$$

What normalization is used for the momentum states? Are the matrix elements of the T-matrix on the right side on-shell?

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d) Write the Lippmann-Schwinger equation for the wave function in coordinate space for a local potential $V = V(\mathbf{r})$. To this end, show first that the free Green's function

$$G^{\pm}(\mathbf{r}', \mathbf{r}; E = k^2/m) = \langle \mathbf{r}|\frac{1}{E - H_0 \pm i\epsilon}|\mathbf{r}'\rangle,$$

is given by

$$G^{\pm}(\mathbf{r}', \mathbf{r}; E = k^2/m) = -\frac{m}{4\pi}\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}.$$

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e) Show that when the T-matrix is evaluated on-shell, it is proportional to the scattering amplitude, $T^{+}(E = k^2/m) = -\frac{1}{4\pi^2 m} f(k,\theta)$, by analyzing the asymptotic form of the Lippmann-Schwinger equation and comparing to

$$\langle \mathbf{r}|\psi_E^{+}\rangle \overset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(k,\theta)\frac{e^{ikr}}{r}\right).$$

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f) Start from the momentum-space partial wave expansion of the potential,

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{2}{\pi}\sum_{l,m} V_l(k',k) Y_{lm}^*(\Omega_{k'}) Y_{lm}(\Omega_k),$$

and a similar expansion of the T-matrix to derive the partial wave version of the Lippmann-Schwinger equation (with the correct factor for the integral):

$$T_l(k',k;E) = V_l(k',k) + \frac{2}{\pi}\int_0^\infty dp\, p^2 \frac{V_l(k',q)T_l(q,k;E)}{E - p^2/m + i\epsilon}.$$

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g) Scattering phase shifts for a square well potential. Calculate the S-wave scattering phase shifts for an attractive square-well potential $V(r) = -V_0\theta(R-r)$ and show that

$$\delta_0(E) = \arctan\left[\sqrt{\frac{E}{E+V_0}}\tan\bigl(R\sqrt{2\mu(E+V_0)}\bigr)\right] - R\sqrt{2\mu E}.$$

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h) Let's consider the analytic structure of the corresponding partial-wave S matrix, which is given by

$$S_0(k) = e^{-2ikR}\frac{k_0\cot k_0R + ik}{k_0\cot k_0R - ik},$$

where $E = k^2/2\mu$ and $k_0^2 = k^2 + 2\mu V_0$. Show that $S_l(k) = e^{2i\delta_l(k)}$ for $l = 0$ is satisfied. Treat $S_0(k)$ as a function of the complex variable k and find its singularities.

Hint. Hint: write $e^{2i\delta} = e^{i\delta}/e^{-i\delta}$.
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i) Bound states are associated with poles on the imaginary axis in the upper half plane. Show that the condition for such a pole here gives the same eigenvalue condition (a transcendental equation) that you would get from a conventional solution to the square well by matching logarithmic derivatives.

Hint. Hint: Define $k = i\kappa$ with $\kappa > 0$ when analyzing such a pole.